Physics 232 – Thermal Waves

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Purpose

In this experiment you will investigate heat flow in a "one-dimensional" rod. You will use Fourier analysis (Fourier transforms) to examine the frequency content of thermal waves has they propagate down the length of a copper rod. From this analysis you will determine the thermal diffusivity of copper. You will also observe the phase shift (or phase velocity) of the thermal oscillations as they propagate down the length of the copper rod. The phase shift data can also be used to extract the thermal diffusivity of copper.

Introduction

Consider a cylindrical rod which has a temperature gradient along its length as in Fig. 1. A heat

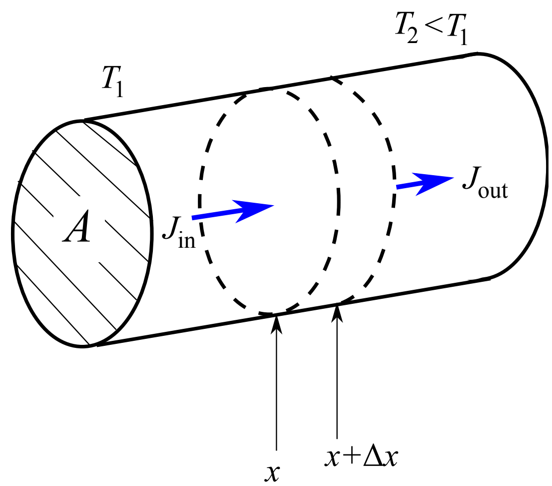

Figure 1: Heat flow in a cylindrical rod.

flux J (heat Q per unit area A per unit time, $[J]=\mathrm{Jm^{-2}s^{-1}}$) flows from the hot end of the rod to the cold end. The constant of proportionality that relates the heat flux and the temperature gradient is the thermal conductivity κ, such that, in one dimension:

$$J = -\kappa \frac{\partial T}{\partial x}. \tag{1}$$

The negative sign indicates that the flow of heat is in the opposite direction of the temperature gradient. The heat capacity is a material property that measures how much heat is required to change the temperature of that material. In terms of the specific heat per unit volume which we denote c_v:

$$V c_v = \frac{\partial Q}{\partial T}, \tag{2}$$

where V is the volume of material. Consider the infinitesimal section of cylinder in Fig. 1 that has volume $A\Delta x$. In this case:

$$\Delta Q = A \Delta x c_v \Delta T. \tag{3}$$

Dividing by a small time interval Δt and taking the limit $\Delta t \to 0$ we can write:

$$\frac{\partial Q}{\partial t} = A \Delta x c_v \frac{\partial T}{\partial t}. \tag{4}$$

Recalling that the heat flux (Eq. 1) is the rate of heat flow per unit area we can express the net heat flow into the disc of the cylinder as:

$$\frac{\partial Q}{\partial t} = A \left[J_{\text{in}} - J_{\text{out}} \right] = A \left[J(x,t) - J(x + \Delta x, t) \right]. \tag{5}$$

Equating Eqns 4 and 5 and dividing by Δx gives:

$$c_v \frac{\partial T}{\partial t} = -\frac{J(x + \Delta x, t) - J(x,t)}{\Delta x} = -\frac{\partial J}{\partial x}, \tag{6}$$

where the second equality holds for infinitesimal Δx. Finally substituting Eq. 1 for J results on the 1-D heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \tag{7}$$

where $\alpha \equiv \kappa / c_v$ is the thermal diffusivity.

Fourier Series

Recall from calculus that any piecewise smooth function can be written in terms of an infinite sum of cosine and sine waves:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{8}$$

where:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx \tag{9a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \tag{9b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx. \tag{9c}$$

See the supplemental information on the course website for a review of the Fourier series.

In this experiment, a switch is used to periodically turn a heater that is attached to one end of a copper rod off and on. This results in a square wave temperature oscillation at that end of the rod (at $x = 0$, for example). You will use Fourier analysis and the heat equation (Eq. 7) to analyze how this temperature oscillation evolves as it propagates along the length of the copper rod.

At any position x, the temperature of the rod is given by $T(x,t) = T_{\mathrm{DC}}(x) + \widetilde{T}(x,t)$ where T_{DC} is the average temperature and $\widetilde{T}$ describes the temperature oscillation. In this experiment we will focus on oscillating part of the temperature. We've already argued that at $x = 0$ the temperature oscillation will be a square wave which, when written as a Fourier series, is:

$$\widetilde{T}(0,t) = \sum_{n\ \mathrm{odd}} \frac{4T_0}{n\pi}\sin\frac{2n\pi t}{\tau}, \tag{10}$$

where the square wave has amplitude T_0 and period τ (τ is used to denote the period since T is reserved for temperature). See the supplemental material for a review of the Fourier series of a square wave.

Now we are in a position to look for a solution to the 1-D heat equation subject to the boundary condition (Eq. 10). The heat equation will not introduce new frequencies so we assume that a solution that will satisfy the boundary condition will be of the form:

$$\widetilde{T}(x,t) = \sum_{n\ \mathrm{odd}} A_n(x)\sin(\omega_n t - k_n x), \tag{11}$$

where we allow for a position-dependent phase factor $k_n x$. The goal is now to determine $A_n(x)$, ω_n, and k_n. Start by evaluating the following derivatives:

$$\frac{\partial \widetilde{T}}{\partial t} = \sum_{n \text{ odd}} A_n(x)\omega_n \cos(\omega_n t - k_n x)$$

$$\frac{\partial^2 \widetilde{T}}{\partial x^2} = \sum_{n \text{ odd}} \left\{ \left[\frac{\partial^2 A_n(x)}{\partial x^2} - A_n(x)k_n^2 \right] \sin(\omega_n t - k_n x) - 2k_n \frac{\partial A_n(x)}{\partial x} \cos(\omega_n t - k_n x) \right\}.$$

Substituting these results into the heat equation and equating the coefficients of the cosine and sine terms results in:

$$-2k_n \frac{\partial A_n(x)}{\partial x} = \frac{\omega_n}{\alpha} A_n(x) \tag{13a}$$

$$\frac{\partial^2 A_n(x)}{\partial x^2} = k_n^2 A_n(x) \tag{13b}$$

Equation 13a can be solved to yield $A_n(x) = B \exp(\omega_n x / 2\alpha k_n)$ which can then be substituted in Eq. 13b. Making this substitution relates k_n to ω_n:

$$k_n = \sqrt{\frac{\omega_n}{2\alpha}}. \tag{14}$$

Returning to Eq. 11 and inserting the solution for $A_n(x)$ gives:

$$\widetilde{T}(x,t) = \sum_{n \text{ odd}} B \exp\left(-\sqrt{\frac{\omega_n}{2\alpha}} x\right) \sin\left(\omega_n t - \sqrt{\frac{\omega_n}{2\alpha}} x\right). \tag{15}$$

Finally, setting $x = 0$ and comparing to the boundary condition given in Eq. 10 sets B and ω_n:

$$B = \frac{4T_0}{n\pi} \tag{16}$$

$$\omega_n = \frac{2n\pi}{\tau} \tag{17}$$

$$\widetilde{T}(x,t) = \sum_{n \text{ odd}} \frac{4T_0}{n\pi} \exp\left(-\sqrt{\frac{\omega_n}{2\alpha}} x\right) \sin\left(\omega_n t - \sqrt{\frac{\omega_n}{2\alpha}} x\right). \tag{18}$$

Notice a couple of features Eq. 18. First, $\widetilde{T}(x,t)$ contains all of the same frequencies ω_n as $\widetilde{T}(0,t)$, but the amplitude of each frequency is attenuated by a *different* exponential factor *and* each frequency has a *different* phase factor. Second, consider what happens when $x \gg \sqrt{2\alpha/\omega_n}$. In this case, the first term in the sum dominates and the temperature oscillation approaches a pure sine wave.

Fourier Transforms

A Fourier transform can be used to extract the amplitude of all frequencies contained in a given signal. That is, the Fourier transform of a signal measured in the time domain [$y(t)$] is the same signal but in the frequency domain [$Y(f)$]. For example, Fig. 2 shows the Fourier transforms of a square wave, a triangle wave, and a sawtooth signal. Notice that these signals contain only integer multiples of the fundamental frequency f_0. The square and triangle waves have only the odd integer multiples, but with different relative amplitudes. The sawtooth wave contain both even and odd multiples of f_0. What would the Fourier transform of a pure sine wave look like?

Pre-lab Assignment

You will use Python to take Fourier transforms of your measured data. One of the Python notebooks in the supplemental material show you how to take Fourier transforms in using a Python module/function. (A separate Python notebook shows you how to convert the measured resistances to temperature and remove the average background signal.)

Your assignment is to modify the Python notebook (called "Thermal waves FFT.ipynb") to take the Fourier transform of the square wave data that is also included as part of the supplemental material. Don't do the resistance→temperature conversion to the square wave data. Just take the Fourier transform and plot it. What is the fundamental frequency of the square wave? What other frequencies are in the Fourier transform? Do the ratio of the amplitudes of the different frequency components agree with what you expect?

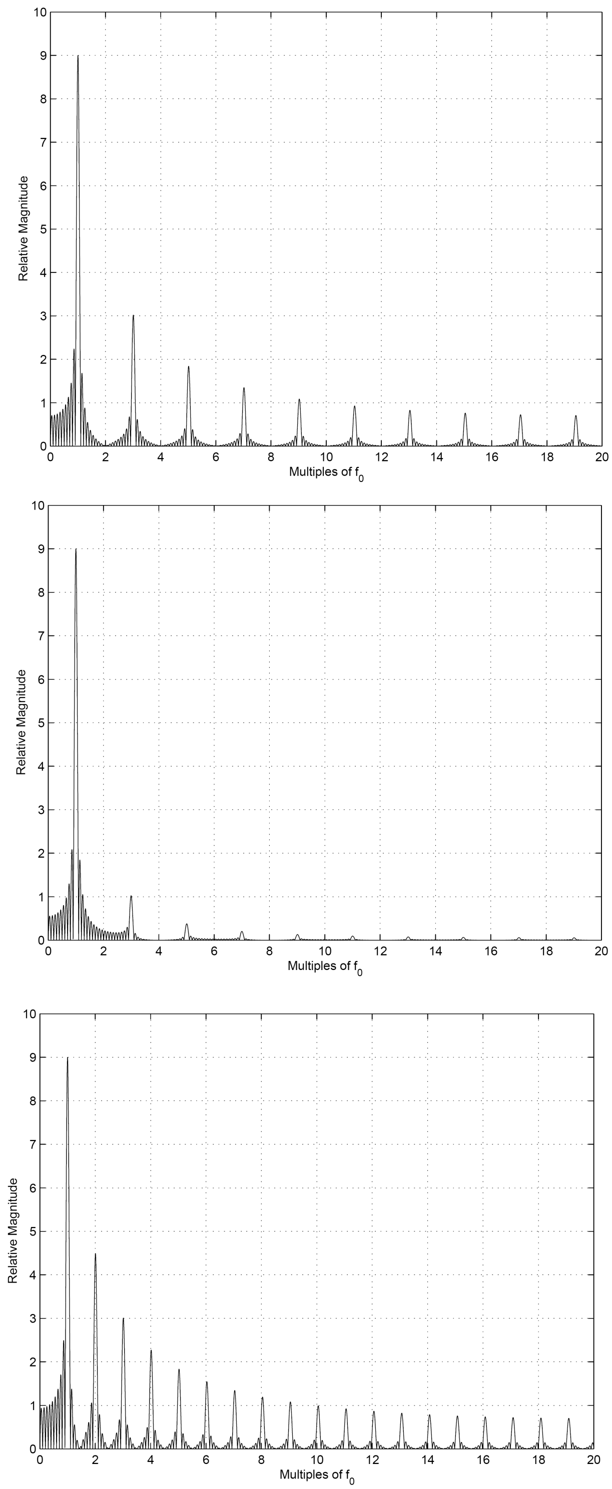

Figure 2: Fourier transforms of a square wave, triangle wave, and sawtooth signal.

Experiment

The apparatus is a long copper rod with holes drilled at various location into which thermometers can be inserted to monitor the temperature. The thermometers are called thermistors and are resistors whose resistance depends sensitively on temperature. The thermistor calibration data is part of the supplemental material. A heater is inserted into one end of the copper rod and it is powered using a variac (variable 60 Hz power supply). Power to this heater is turned off and on periodically using a relay (essentially a switch) controlled by a function generator.

Start with the thermistors in the three holes closest to the heater. Set the variac to ≈ 90 V (DO NOT TOUCH THE TERMINALS OF THE RELAY) and the function generator to a ≈ 0.005 Hz square wave. You will use three Agilent 34401A multimeters to measure the resistance of the three thermistors. The multimeters will be controlled using a computer program written using LabVIEW. The program will write the data to a file which you can then analyze using Python. Set the program to record the thermistor resistances every two seconds.

Analysis

We want to analyze the temperature oscillation and not the transient behaviour. Remove the part of the data over which the average temperature changes significantly. Next subtract the average temperature from the data. You should now have a temperature oscillation that is centred on zero.

Take the Fourier transform of the data so that you can examine the frequency profile of the temperature oscillations. Plot the Fourier transform and observe the different frequencies contained in the signal and their relative amplitudes. How do these features change with distance x from the heater? Use the amplitude of the fundamental frequency as a function of x to determine the thermal diffusivity α of copper. Your analysis needs error estimates and units! Does your final value for α agree with the expected value? Could you also determine α from the phase of the signals as a function of x?